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POSSIBLE EXTENSIONS OF THE 4-D SCHWARZSCHILD HORIZON IN THE BRANE WORLD

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Abstract

We show that, in the absence of matter in the bulk, the Einstein equations and the Gauss-normal form of the metric place stringent restrictions on the form of the event horizon. As a consequence, the off-brane extension of the standard 4-D Schwarzschild horizon, in the coordinate system based on the brane can only be of a tubular shape, instead of a pancake shape. When the coordinate system is based on the AdS_5 horizon, such a tubular horizon is absent.

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The Randall-Sundrum model [1] consists of a three-dimensional brane embedded in a five-dimensional spacetime which is asymptotically anti-de Sitter. The Gauss-normal form of the metric in this space is

$$ds^2 = e^{-2\kappa|y|}\bar{g}_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{1}$$

with the four-dimensional metric $\bar{g}_{\mu\nu}$ being asymptotically flat. The asymptotic metric with $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ satisfies the vacuum Einstein equations

$$R_{mn} - \frac{1}{2}Rg_{mn} - \Lambda g_{mn} = 0, \tag{2}$$

with the cosmological constant $\Lambda = -6\kappa^2$. Here and throughout the paper, we adopt the convention that the Greek indices take values 0-3 and the Latin indices 0-4. The 5-D Einstein equations outside the source which is localized on the *physical brane* can be rewritten as

$$R_{\mu\nu} - 4\kappa^2 g_{\mu\nu} = 0, \quad R_{y\mu} = 0, \quad R_{yy} - 4\kappa^2 = 0, \tag{3}$$

with $g_{\mu\nu} = e^{-2\kappa|y|}\bar{g}_{\mu\nu}$. In the absence of nonzero modes (i.e., for a y independent metric $\bar{g}_{\mu\nu}$), the five-dimensional Einstein equations with negative cosmological constant reduce to the four-dimensional Einstein equations in vacuum without a cosmological constant. Consequently, the Randall-Sundrum scenario can be extended, provided the metric $\bar{g}_{\mu\nu}$ is Ricci flat, i.e., it corresponds to any vacuum solution of 4-D General Relativity. The Schwarzschild solution [2], the source of which is a mass line extending infinitely in the y-direction, constitutes an example:

$$ds^2 = e^{-2\kappa|y|}\Big[-(1-\frac{2GM}{r})dt^2 + \frac{1}{1-\frac{2GM}{r}}dr^2 + r^2 d\Omega^2\Big] + dy^2. \tag{4}$$

To the best of our knowledge, a physical black hole solution of the Einstein equations in the Randall-Sundrum framework –a gravitational field generated by a mass point on the *physical brane*–has not appeared except in $3+1$ dimensions (a two-dimensional brane embedded in a four-dimensional spacetime) [3]. An approximate solution of the Einstein equations with a static, spherically symmetric matter distribution on the *physical brane* to second order in the gravitational coupling was previously obtained [4]. In the coordinates straight with respect to the brane [5], this expression is valid in a region far away from the source both on the *physical brane* and in the y-direction. When it is confined on the brane, $y = 0$, it coincides with the standard Schwarzschild form to the second order in the gravitational coupling. In the coordinates straight with respect to the AdS_5 horizon, the solution is valid for large r or large and positive y, but the brane is bent in the direction of negative y.

It is generally believed that 4-D general relativity is recovered beyond the weak coupling expansion on the *physical brane* for large κ [6]. Giddings, Katz and Randall reproduced 4-D linear gravity on the *physical brane* for large κ [7]. By combining the

Schwarzschild metric on the *physical brane* and the profile of the linear gravitational potential off the *physical brane*, they conjectured a pancake shaped horizon for the physical black hole. In this letter, we attempt to determine the form of the horizon of a physical black hole, which when confined to the *physical brane* reproduces the standard 4-D Schwarzschild black hole solution. We find that the Einstein equations together with a Gauss-normal form of the metric imply particular types of horizons but the pancake shape does not belong to these types.

The most general metric in $D = 4+1$ dimensions produced by a static, spherically symmetric matter distribution on the *physical brane* has the following form:

$$ds^2 = e^{-2\kappa|y|}(-e^a dt^2 + e^b dr^2 + e^c r^2 d\Omega^2) + dy^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the solid angle on S^2 and a, b and c are functions of r and y. Substituting the metric (5) into equations (3), we obtain the following nonzero components of the Einstein equation outside the source:

$$\begin{aligned} R_{tt} + 4\kappa^2 e^{-2\kappa y + a} &= \frac{1}{2}e^{a-b}\Big[-a'' - \frac{2}{r}a' + \frac{1}{2}a'(-a' + b' - 2c')\\ &\quad - \frac{1}{2}e^{-2\kappa y + a}\Big[\ddot{a} - 5\kappa\dot{a} - \kappa\dot{b} - 2\kappa\dot{c} + \frac{1}{2}\dot{a}(\dot{a} + \dot{b} + 2\dot{c})\Big] = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} R_{rr} - 4\kappa^2 e^b &= \frac{1}{2}a'' + c'' - \frac{1}{r}b' + \frac{2}{r}c' + \frac{1}{4}a'(a' - b') - \frac{1}{2}c'(b' - c')\\ &\quad + \frac{1}{2}e^{-2\kappa y + b}\Big[\ddot{b} - 5\kappa\dot{b} - \kappa\dot{a} - 2\kappa\dot{c} + \frac{1}{2}\dot{b}(\dot{a} + \dot{b} + 2\dot{c})\Big] = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} R_{\theta\theta} - 4\kappa^2 r^2 e^c &= \frac{1}{2}r^2 e^{c-b}\Big[c'' + \frac{4}{r}c' + \frac{a' - b'}{r} + c'^2 + \frac{1}{2}(a' - b')c'\Big]\\ &\quad + \frac{1}{2}r^2 e^{-2\kappa y + c}\Big[\ddot{c} - \kappa(\dot{a} + \dot{b}) - 6\kappa\dot{c} + \frac{1}{2}\dot{c}(\dot{a} + \dot{b} + 2\dot{c})\Big] + e^{c-b} - 1 = 0 \end{aligned} \tag{8}$$

$$\begin{aligned} R_{yy} - 4\kappa^2 &= \frac{1}{2}(\ddot{a} + \ddot{b} + 2\ddot{c}) - \kappa(\dot{a} + \dot{b} + 2\dot{c}) + \frac{1}{4}(\dot{a}^2 + \dot{b}^2 + 2\dot{c}^2) = 0\\ R_{ry} = R_{yr} &= \frac{1}{2}\Big[\dot{a}' + 2\dot{c}' - \frac{2}{r}(\dot{b} - \dot{c}) + \frac{1}{2}a'(\dot{a} - \dot{b}) - c'(\dot{b} - \dot{c})\Big] = 0, \end{aligned} \tag{9}$$

where the prime denotes a partial derivative with respect to r and the dot denotes a partial derivative with respect to y. These equations apply to the positive side of the brane, $y > 0$, the corresponding equations to the negative side of the brane, $y < 0$, are obtained by switching the sign of κ.

Let us formulate the concept that would define an event horizon in the 5-D brane world. We assume that there exists a solution of the 5-D Einstein equations which satisfies the Israel matching condition across the brane [8] and maintains the Lorentzian signature in a certain region, $\mathcal{P}$, of the parametric $r - y$ plane. We call $\mathcal{P}$ the physical region. The physical region may or may not cover the entire $r - y$ plane. An example of a physical region that does not cover the entire parametric space is the AdS C-metric discussed in [3]. We also assume that there exists an asymptotic region $\mathcal{A}$ within $\mathcal{P}$, where the metric components e^a, e^b and e^c are well approximated by linear gravity and the functions a, b and c are well behaved beyond $\mathcal{A}$. As we have seen, this is the case for both brane-based coordinates and for AdS_5 horizon- based coordinates. Starting from $\mathcal{A}$, we trace all possible light rays given by $ds^2 = 0$ or more specifically by

$$dt^2 = e^{b-a}dr^2 + e^{c-a}r^2 d\Omega^2 + e^{2\kappa y-a}dy^2 \tag{10}$$

until we come to a point from which the light can not propagate forward in certain spatial direction. By continuity, the union of these points forms a 4-D hypersurface, $\mathcal{H}$, which we refer to as an event horizon. We shall consider the case that $\mathcal{H}$ lies within (not coincide with the border of) $\mathcal{P}$.

To prevent light propagation, some of the coefficients on the right hand side of (10) need to become sufficiently divergent so that an increment of the corresponding spatial coordinate would take infinite amount of time, t. Therefore, we can determine the form of the horizon by finding the locus of the logarithmic singularities of the functions a, b and c. This locus has to be consistent with the Einstein equations (6)-(9).

We denote the locus of the logarithmic singularities of a, b and c by $H(r, y) = 0$ and an arbitrary point on it by $P(r_0, y_0)$. The unit normal vector $\vec{n}$ and the unit tangent vector $\vec{t}$ at P on $r - y$ plane are

$$\vec{n} = (\cos\alpha, \sin\alpha), \qquad \vec{t} = (-\sin\alpha, \cos\alpha), \tag{11}$$

where

$$\cos\alpha = \frac{1}{\Delta}\left(\frac{\partial H}{\partial r}\right)_P, \qquad \sin\alpha = \frac{1}{\Delta}\left(\frac{\partial H}{\partial y}\right)_P \tag{12}$$

and

$$\Delta = \sqrt{\left(\frac{\partial H}{\partial r}\right)_P^2 + \left(\frac{\partial H}{\partial y}\right)_P^2}. \tag{13}$$

Let's consider a point $Q(r, y)$ in the neighbourhood of P and let's transform the coordinates into

$$\begin{aligned} \xi &= (r - r_0)\cos\alpha + (y - y_0)\sin\alpha \\ \eta &= -(r - r_0)\sin\alpha + (y - y_0)\cos\alpha. \end{aligned} \tag{14}$$

As $\xi \to 0$ and $\eta \to 0$, we expect that

$$e^a \simeq A\xi^{n_a}, \qquad e^b \simeq B\xi^{n_b}, \qquad e^c \simeq C\xi^{n_c} \tag{15}$$

where n_a, n_b and n_c are integers due to the reality requirement of the metric components on both sides of the horizon, and A, B and C are numerical constants. The integers n_a, n_b and n_c could not all be zero, otherwise there would be no singularity at the point P.

The Lorentzian signature in the physical region forbids e^c and the metric determinant $e^{-8\kappa y+a+b+2c}r^4\sin^2\theta$ from changing their signs. These considerations restrict both n_c and $n_a+n_b+2n_c$ to be even. By continuity, these exponents are maintained along the entire H. Therefore,

$$a \simeq n_a \ln\xi, \qquad b \simeq n_b \ln\xi, \qquad c \simeq n_c \ln\xi. \tag{16}$$

Taking into account that

$$\begin{aligned} \frac{\partial}{\partial r} &= \cos\alpha\frac{\partial}{\partial\xi} - \sin\alpha\frac{\partial}{\partial\eta} \\ \frac{\partial}{\partial y} &= \sin\alpha\frac{\partial}{\partial\xi} + \cos\alpha\frac{\partial}{\partial\eta}, \end{aligned} \tag{17}$$

their derivatives behave as

$$\dot{a} \simeq \frac{n_a}{\xi}\sin\alpha, \qquad \dot{b} \simeq \frac{n_b}{\xi}\sin\alpha, \qquad \dot{c} \simeq \frac{n_c}{\xi}\sin\alpha \tag{18}$$

and

$$a' \simeq \frac{n_a}{\xi}\cos\alpha, \qquad b' \simeq \frac{n_b}{\xi}\cos\alpha, \qquad c' \simeq \frac{n_c}{\xi}\cos\alpha. \tag{19}$$

We observe that the derivatives of a, b, c become singular as Q approaches P. As a result of substituting (18) and (19) into equations (6)-(9), the cancellation of the leading sigularities in $R_{yy}-4\kappa^2$ and in R_{ry}, $O(\frac{1}{\xi^2})$, leads to two conditions on n_a, n_b, n_c and α:

$$E\sin^2\alpha = 0, \qquad F\sin\alpha\cos\alpha = 0, \tag{20}$$

with

$$E = -n_a - n_b - 2n_c + \frac{1}{2}(n_a^2 + n_b^2 + 2n_c^2) \tag{21}$$

and

$$F = -n_a - 2n_c + \frac{1}{2}n_a(n_a - n_b) - n_c(n_b - n_c). \tag{22}$$

There are three types of solutions to be analyzed.

Type I: $E \neq 0$.

The only solution is $\sin\alpha = 0$, which implies a tube shaped H:

$$\left(\frac{\partial H}{\partial y}\right)_P = 0. \tag{23}$$

The standard Schwarszchild horizon of a physical black hole belongs to this type. In the coordinates straight with respect to the brane, the recovery of 4-D general relativity on the brane for $\kappa GM >> 1$ implies that

$$ds^2 \simeq -(1-\frac{2GM}{r})dt^2 + \frac{dr^2}{1-\frac{2GM}{r}} + r^2 d\Omega^2 \tag{24}$$

for $y = 0$. The standard Schwarzschild horizon at $r \simeq 2GM$ corresponds to the exponents $n_a = 1$, $n_b = -1$ and $n_c = 0$. Therefore, since the integer combination $E \neq 0$, the off-brane extention of the horizon is a tube as is shown in Fig. 1a. It either extends to infinity in the y direction, similar to the horizon of the black cigar solution of Chamblin-Hawking-Reall (though the explicit forms of the solution are different), or it terminates at the border of the physical region, similar to the example in [3]. This rules out the possibility of a pancake shaped Schwarzschild horizon in 5-D. On the other hand, in the coordinates straight with respect to the AdS_5 horizon, the validity of linear gravity for large r or large positive y excludes the tubular form of the horizon completely, as is shown in Fig. 1b. Because of the brane bending in the negative y-direction and the failure of the linear approximation there, we are unable to rule out the possibility of horizons of the subsequent two types in this coordinate system.

Before analyzing the next two types, we notice that the solutions to $E = 0$ correspond to points with integer coordinates lying on an oblate spheroid with axis 2 and $\sqrt{2}$. There are only twelve of them, and we should exclude the solutions with odd n_c or odd $n_a + n_b + 2n_c$ and the ones that make none of e^{b-a}, e^{c-a} and e^{-a} divergent. Consequently, we have

Type II: $E = 0$ but $F \neq 0$.

The only solutions in this case are $(n_a, n_b, n_c) = (2,\ 2,\ 0)$ or $(2,\ 2,\ 2)$ with either $\sin\alpha = 0$ or $\cos\alpha = 0$. The latter implies a horizon parallel to the plane $y = 0$ and it may exist in the coordinates straight with respect to the AdS_5 horizon for a physical black hole.

Type III: $E = 0$ and $F = 0$.

The qualified solutions for (n_a, n_b, n_c) are $(2,\ 0,\ 0)$ and $(2,\ 0,\ 2)$. They are also consistent with the other components of the Einstein equations. We have not found yet any restrictions on the shapes of the corresponding horizons. The first solution of the integer triplet corresponds to the isotropic coordinates of a black hole ($n_a = 2, n_b = n_c = 0$), which can be obtained through a $y -$ *independent* coordinate transformation from the standard metric. It can be shown, however, that a $y -$ *dependent* and Gauss-normal form preserving transformation that leaves the brane intact does not exist. Therefore, this particular horizon cannot be transformed into a pancake shaped one.

For a general Gauss-normal form of the metric (1), we may define the 4×4 matrix of $\bar{g}_{\mu\nu}$ by $\mathcal{G}$, and the $R_{yy} - 4\kappa^2$ equation can be written as

$$R_{yy} - 4\kappa^2 = \frac{1}{2}\Big(\frac{\partial}{\partial y}e^{-2\kappa y}\frac{\partial}{\partial y}\ln(-\det\mathcal{G})\Big) + \frac{1}{4}\mathrm{tr}\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial y}\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial y} = 0. \tag{25}$$

Now our statements regarding the Schwarszchild horizon can be generalized to a spinning black hole located on the brane $y = 0$. Assume that the 4-D Kerr metric [9] has been recovered on the brane for sufficiently large κ, then we have

$$ds^2 \simeq -\frac{\Delta}{\rho^2}(dt - j\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+j^2)d\phi - jdt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \qquad (26)$$

where $\Delta = r^2 - 2GMr + j^2$ and $\rho^2 = r^2 + j^2\cos^2\theta$ with M the mass and j the angular momentum per unit mass. A horizon corresponds to the solutions of $\Delta = 0$, that might have two solutions or none. It is straightforward to show that the metric determinant $\det\mathcal{G} = -(r^2 + j^2\cos^2\theta)^2\sin^2\theta$ and its logarithm are free from the horizon singularities. On the other hand, the matrix $\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial r}$ contains a $\frac{1}{\Delta}$ singularity at the horizon. If the 5-D extension of this horizon was bent towards or away from the y axis, this singularity would be shared by the matrix $\mathcal{G}^{-1}\frac{\partial\mathcal{G}}{\partial y}$ off the brane. This is again forbidden by the Einstein equation $R_{yy} - 4\kappa^2 = 0$.

Having a pancake shaped horizon as the 5-D extension of the Schwarszchild horizon is physically implausible. If this were the case, somewhere off the brane and near the horizon, we would expect that

$$e^a \sim y - y_c, \qquad e^b \sim \frac{1}{y - y_c}, \qquad (27)$$

with y_c a function of r. It follows then from equation (10) that it would take a finite amount of time for a light beam coming out of the horizon to propagate in the y-direction. Consequently the black hole would appear to be luminating in the y-direction.

The conclusions we have reached above depend on the form of the metric, equation (1), and the Einstein equations. We might try to relax the Gauss normal form of the metric. In the case of a spherical black hole, we might consider the metric

$$ds^2 = e^{-2\kappa|y|}(-e^u dt^2 + e^v dr^2 + r^2 d\Omega^2) + e^f dy^2, \qquad (28)$$

where u, v, f are functions of r and y. The equation $R_{yy} - 4\kappa^2 = 0$ then becomes

$$\ddot{u} + \ddot{v} + \frac{1}{2}(\dot{u}^2 + \dot{v}^2) + e^{2\kappa y + f - v}(f'' + \frac{1}{2}f'^2) = 0. \qquad (29)$$

Let's assume that an approximate Schwarszchild metric can be recovered on the brane, which is located at $y = 0$ and that the 5-D extension of the horizon is given by $H(r, y) = 0$. If we consider an arbitrary point on it, $P(r_0, y_0)$, with the variable ξ defined as before, we have

$$u \sim \ln\xi \qquad v \sim -\ln\xi \qquad f \sim n_c \ln\xi \qquad (30)$$

with n_c being an even integer. Then,

$$\ddot{u} + \ddot{v} + \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \sim \frac{1}{2\xi^2}\sin^2\alpha \qquad (31)$$

and

$$e^{2\kappa y+f-v}(f''+\frac{1}{2}f'^2)\sim(-n_c+\frac{1}{2}n_c^2)\xi^{n_c+1-2}\cos^2\alpha. \tag{32}$$

For $n_c<-1$, (32) represents the leading singularity of the equation (29), the only solution is $\cos\alpha=0$ and this horizon will not join the Schwarszchild horizon on the brane. If $n_c>-1$, (31) represents the leading singularity, and the only solution is $\sin\alpha=0$. Therefore, the 5-D extension of Schwarszchild horizon is again a tube.

The existence of a 5-D extension of a 4-D Schwarzschild horizon for a physical black hole relies on the recovery of 4-D general relativity on the brane for large κ. The rigorous statements we made on the possible shapes of the horizon, however, are independent of the value of κ, since the terms of the first derivative with respect to y in equations (6)-(9) do not contribute to the leading singularity of a horizon. In the following, we illustrate various types of horizons for a 5-D Schwarzschild metric ($\kappa=0$), i. e.,

$$ds^2=-\Big(1-\frac{l^2}{R^2}\Big)dt^2+\frac{dR^2}{1-\frac{l^2}{R^2}}+R^2(d\chi^2+\sin^2\chi d\Omega^2), \tag{33}$$

where l is the Schwarzschild radiu, R is the radial coordinate, and χ is a polar angle on S^3, $d\Omega$ is the solid angle on S^2. If we introduce cylindrical coordinates $r'=R\sin\chi$ and $y'=R\cos\chi$, we find

$$ds^2=-\Big(1-\frac{l^2}{R^2}\Big)dt^2+\frac{l^2}{1-\frac{l^2}{R^2}}\frac{(r'dr'+y'dy')^2}{R^2}+dr'^2+r'^2d\Omega^2+dy'^2, \tag{34}$$

and we have a circular horizon on $r'-y'$ plane at the expense of introducing off-diagonal terms of the metric. To enforce the Gauss-normal form of the metric, consider the transformation

$$R=R(r,y)\qquad \chi=\chi(r,y). \tag{35}$$

The functions $R(r,y)$ and $\chi(r,y)$ have to satisfy the conditions

$$\begin{aligned}\frac{R^2}{R^2-l^2}\Big(\frac{\partial R}{\partial y}\Big)^2+R^2\Big(\frac{\partial\chi}{\partial y}\Big)^2&=1\\ \frac{1}{R^2-l^2}\frac{\partial R}{\partial r}\frac{\partial R}{\partial y}+\frac{\partial\chi}{\partial r}\frac{\partial\chi}{\partial y}&=0.\end{aligned} \tag{36}$$

There is a simple solution, $R=\sqrt{y^2+l^2}$ and χ an arbitrary function of r only, which results in a Gauss-normal form of the metric (34), i. e.,

$$ds^2=-\frac{y^2}{y^2+l^2}dt^2+(y^2+l^2)\Big(\frac{d\chi}{dr}\Big)^2dr^2+(y^2+l^2)\sin^2\chi d\Omega^2+dy^2. \tag{37}$$

The horizon, $y=0$, then becomes of type III. This transformation, however, is singular on the horizon, more specifically the jacobian $\frac{\partial(R,\chi)}{\partial(r,y)}$ is zero. If a solution to equations (36) that is nonsingular at the horizon could be found, then we would have a horizon of type I, since $\lim_{R\to 0^+}\frac{\partial R}{\partial y}=0$ in any case in order to balance the first equation of (36) and the nonvanishing jacobian demands that $\lim_{R\to 0}\frac{\partial R}{\partial r}\neq 0$.

In summary, we have found that the vacuum Einstein equations together with a Gauss-normal form of the metric place fairly stringent restrictions on the form of the event horizon. Our statements are based on the cancellation of the leading singularities in the Einstein tensor. The suggested pancake shaped extension of the 4-D Schwarzschild horizon cannot exist in the Gauss-normal form of the metric. In the coordinate system based on the brane, the event horizon has a tubular shape extending possibly to infinity while in the coordinate system based on the AdS_5 horizon, the region of validity of linear gravity excludes this possibility. The horizon then might belong to either Type II or Type III. Although the exact solution of a physical black hole in the Randall-Sundrum framework is difficult to obtain, our observation might provide hints towards it. It is also interesting to explore the metric of a physical black hole numerically. We hope to report our progress towards this direction in the near future.

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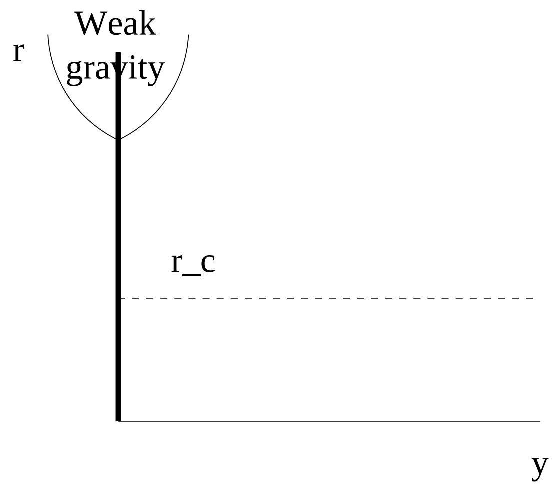

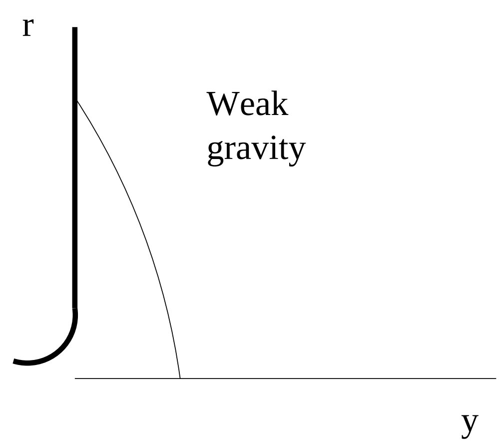

A physical black hole viewed in two coordinate systems:

a) In the coordinates based on the brane, where the thick line denotes the brane and the dashed line the off-brane extension of the 4-D Schwarzschild horizon. The Schwarzschild radius is $r_c = 2GM(1 + O(\frac{1}{\kappa^2 G^2 M^2}))$ and the mass point is located at $r = y = 0$.

b) In the coordinates based on the AdS_5 horizon, where the thick line denotes the bent brane.